Final Exam 2023
Mathematics – Basic course

Classes:	4e
Duration:	4 hours
Number of pages: (not counting this cover)	6 pages

Content: Final exams 2023, Mathematics (written), Basic course

Problem 1: Calculus	17	points
Problem 2: Vector geometry	16	points
Problem 3: Calculus	12	points
Problem 4: Stochastics	14	points
Problem 5: Calculus	9	points

Instructions: Please start solving each question **at the top of a new page!**

Resources: Mathematics Compact formulary (German or English)
(Adrian Wetzel, ISBN 978-3-9523907-7-1)
Calculator TI-30X Pro MultiView or TI-30X Pro MathPrint
English dictionary

Scale: Total number of points: 68
The number of points attainable in each question is stated.
You do not need to attain the full number of points
in order to have mark 6.

Please make sure you have got all documents listed above before starting to solve the questions. If you think that something is missing, inform the supervisor **immediately!**
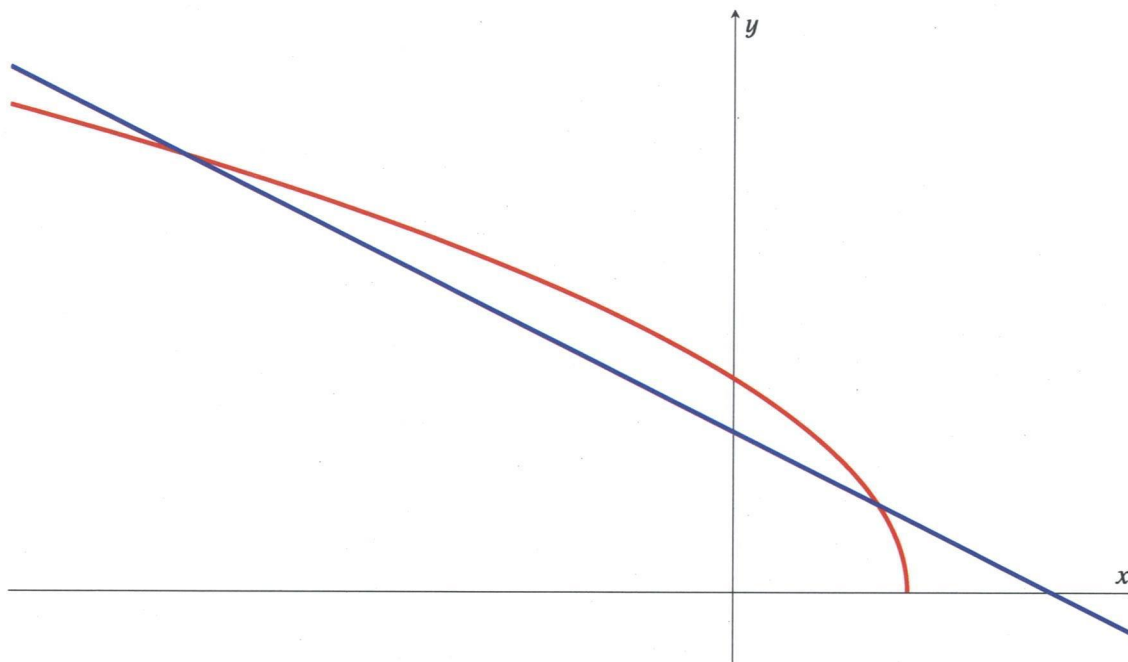
Problem 1   2 + 2 + 3 + 4 + 3 + 3 = **17 points**

Given are the functions f and g with

$$f(x) = 6\sqrt{6-x} \quad \text{and}$$
$$g(x) = -x + 11$$

The graphs of both functions are drawn in the diagram below. Take into account that the scaling on the x-axis is different from the one on the y-axis.

a) Determine the zeros of both functions.

b) Find the angle between the graph of g and the y-axis.

c) Calculate the points of intersections of the graphs of both functions.

d)   (i) Find the point P where the tangent t to the graph of f is parallel to the line g.

   (ii) Determine the equation of this tangent t.

e) The graphs of f and g and the x-axis are the boundaries of a region lying in the first quadrant. Calculate its area.

f) In the interval starting at a lower bound a and ending at the zero of f, the graph of f and the x-axis are the boundaries of a region. This region is rotated around the x-axis to create a solid of revolution with volume $1800 \cdot \pi$. Determine the parameter a.

Problem 2 $2 + 2 + 2 + 2 + 3 + 5 = $ **16 points**

A canopy (Vordach) $ABCD$ is situated at a wall of a house (cf. drawing below).

a) For the canopy $\overrightarrow{AB} = \overrightarrow{DC}$ holds.

   Calculate the coordinates of the vertex B using the information from the drawing.

b) Show that the canopy is a rectangle and calculate its area.

c) The canopy lies in the plane ε_1.

   Calculate a Cartesian equation of ε_1.

   (For control: $\varepsilon_1: x + 2y + 5z - 31 = 0$. You are not allowed to use the equation given here to solve this subproblem.)

d) Calculate the angle of inclination between the canopy and the x-y-plane (ground plane).

e) At midday the sunlight falling on the roof is parallel to the vector $\vec{v} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$ and casts a shadow.

   Determine using a calculation whether the shadow of the point C is visible on the wall of the house. This wall lies in the plane $\varepsilon_2: x + 2y + 0z - 16 = 0$.

f) A little later in the day a resident of the house looks out of a window (cf. drawing) onto the canopy possessing a reflecting surface. She is blinded by a sunbeam reflected by the canopy. The blinded eye of the resident lies at the point $P(4\,|\,6\,|\,15)$.

   Because it's already afternoon the direction of the sunlight is now parallel to the vector $\vec{w} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.
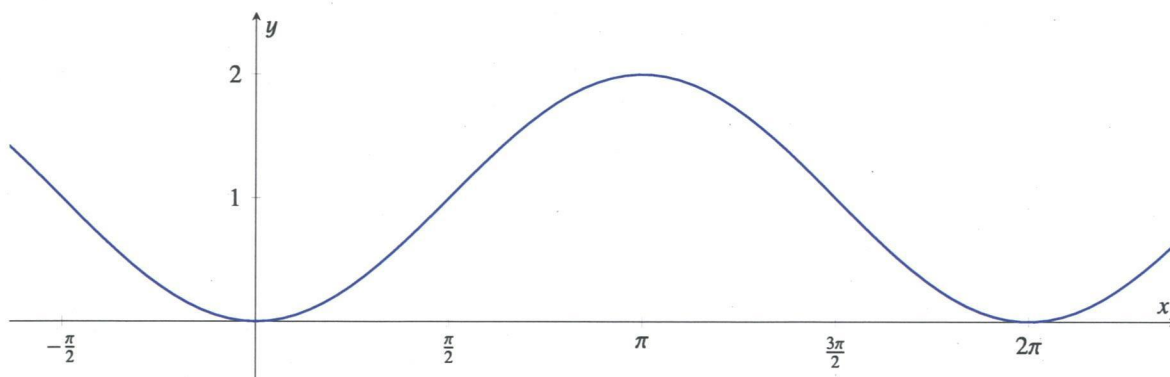
   Calculate the point of reflexion R of the sunlight on the canopy.
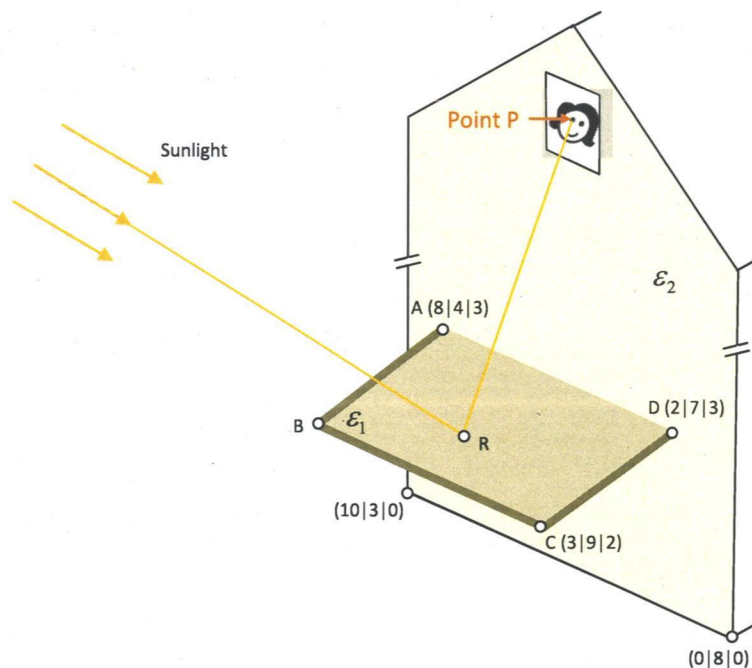
Problem 3

$1.5 + 2.5 + 2 + 2 + 2 + 2 = \textbf{12 points}$

Given is the function f with $f(x) = 1 - \cos x$.

a) The line g is a tangent to the graph of f touching it at $x = \dfrac{\pi}{6}$.

Calculate the slope of the line g.

b) Calculate the area of the region between the graph of f and the x-axis between $x = 0$ and $x = \dfrac{\pi}{2}$ (**exact result**).

c) A random coordinate u with $0 \leq u \leq \dfrac{\pi}{2}$ is chosen.

Calculate the probability that the point $P(u\,|\,0.5)$ lies above the graph of f.

Consider the functions $f_a(x) = 1 - \cos(x) + a \cdot \cos(3x)$.

d) For which value of the parameter a does the graph in the diagram below result?

e) For which value of a does the graph of f_a possess a horizontal tangent at $x = \dfrac{\pi}{2}$?

f) For which values of a does $f_a''(0) > 0$ hold?

$P(\pi|1.5)$
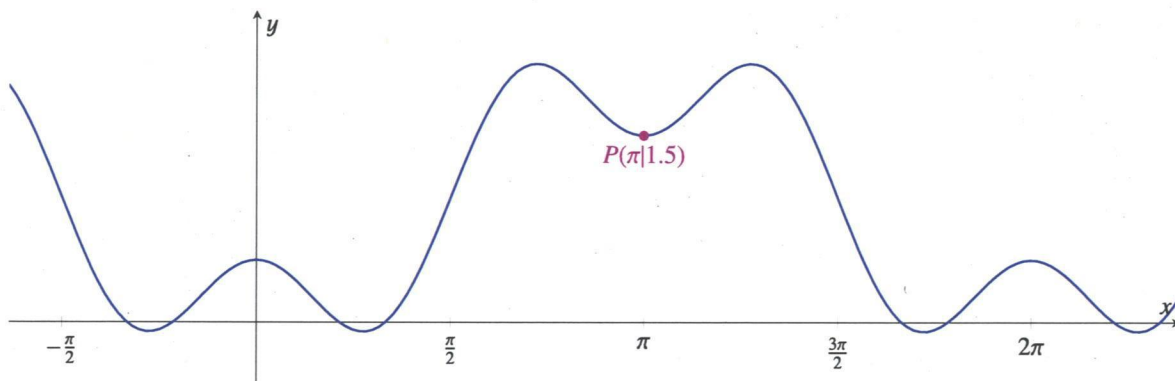
Problem 4

This problem consists of two independent parts 4.1 and 4.2.

Problem 4.1 $1 + 1 + 3 + 2 + 3 = \mathbf{10\ points}$
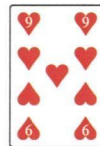
Anna and Basil have found the following 5 playing cards:

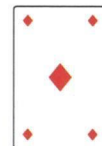| 7 of hearts	7 of spades	9 of hearts	9 of spades	Ace of diamonds
value of the card: 7	value of the card: 7	value of the card: 9	value of the card: 9	value of the card: 11

a) First Anna draws four times one of the five cards and puts it back.

Calculate the probability that she never draws the ace.

Now they shuffle the 5 cards and distribute them. Anna gets 2 cards and Basil 3.

b) Show that there are 10 possibilities for the two cards that Anna gets.

c) Anna calculates the sum of the values of the two cards she gets.

What is the expected value for this sum?

d) Two of the three cards that Basil gets have the same value. Taking this into account, what is the probability that one of the two cards Anna gets is the ace?

e) Anna and Basil decide to play a game with the following rules:

  – First the cards are shuffled and distributed. Anna gets 2 cards and Basil 3.

  – In the first move of the game, Anna draws one of Basil's cards.
    In the second move, Basil draws one of Anna's cards.
    In the third move, Anna draws again one of Basil's cards and so on.

  – As soon as one of them has two cards with the same value (in other words the two sevens or the two nines), he or she is allowed to put down these two cards and they are not longer in the game. As soon as one of them has no cards left, the game ends and the person without cards wins.

Anna got the cards heart 7 and spade 9.
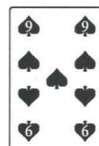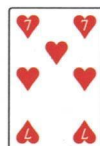
After two moves of the game, either i) Anna wins or ii) Basil wins or iii) they are back in the same situation as before the first move.

Calculate the probabilities for the cases i), ii) and iii).

Problem 4.2 $2 + 2 = \textbf{4 points}$

In a game of cards similar to problem 4.1 Anna and Basil want to decide with the toss of a coin who is allowed to begin the next round of the game (and has a slight advantage). They determine that in case head appears Anna starts and in case tail appears Basil starts.

After a few rounds of the game Anna gets the impression that Basil is allowed to start too many times. Is the coin manipulated against her favour? She decides to conduct a statistic. In the following 12 rounds Basil is allowed to start 8 times.

a) Calculate the probability that at least 8 of 12 tosses are tails in case the coin is fair.

b) Anna decides to continue conducting her statistic.

She wants to conclude with a probability of 95% that the coin shows tail too often.

At least how many times in 20 rounds does she have to toss tail to be able to conclude this?

Problem 5 $1.5 + 2.5 + 5 =$ **9 points**

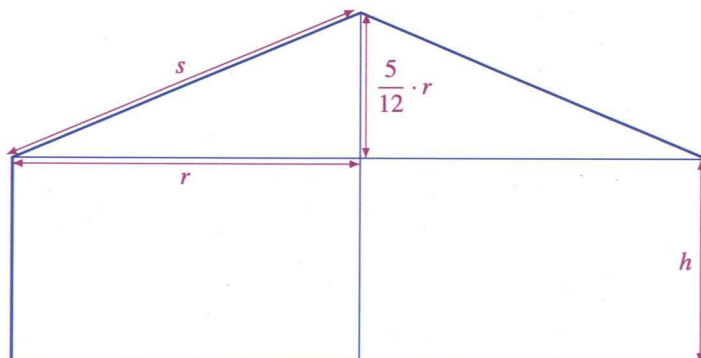The yurt is the traditional tent of the nomads in Mongolia.

A mongolian yurt can be considered as a cylinder with a conical roof on top.

For this type of yurts the height of the cone equals approximately

$$h_{\text{cone}} = \frac{5}{12} \cdot r$$

(cf. drawing to the right).

a) A yurt has a radius of 3 m and an edge height (equals the height h of the cylinder) of 1.8 m. Calculate its volume.

b) Show that the surface of the yurt (without the floor) equals

$$A = \frac{13}{12}\pi r^2 + 2\pi rh$$

c) Mongolian yurts have to withstand extreme weather conditions. To make the loss of energy as small as possible, their surface at a given volume has to be as small as possible.

A yurt has the volume of $V = 50$ m^3. Calculate its radius r and the height h of the cylinder in case the yurt's surface (without floor) is as small as possible.

Calculate the minimum surface as well.

Hint: You don't have to examinate the type of the extremum and the boundary values of the domain.